Optimal shape and location of sensors or actuators in PDE models

Y. Privat, E. Trélat\textsuperscript{1}, E. Zuazua

\textsuperscript{1}Univ. Paris 6 (Labo. J.-L. Lions) et Institut Universitaire de France

SIAM Conference on Analysis of Partial Differential Equations, 2015
What is the best shape and placement of sensors?

- Reduce the cost of instruments.
- Maximize the efficiency of reconstruction and estimations.
The observed system may be described by:

- wave equation \[\partial_{tt}y = \triangle y\]
- Schrödinger equation \[i\partial_t y = \triangle y\]
- general parabolic equations \[\partial_t y = Ay\] (e.g., heat or Stokes equations)

in some domain \(\Omega\), with either Dirichlet, Neumann, mixed, or Robin boundary conditions.

For instance, when dealing with the heat equation:

*What is the optimal shape and placement of a thermometer?*
Waves propagating in a cavity:

\[
\begin{align*}
\partial_{tt} y - \Delta y &= 0 \\
y(t, \cdot)_{\partial\Omega} &= 0
\end{align*}
\]

Observable

\[
y(t, \cdot)_{|\omega}
\]

Observability inequality

The observability constant \( C_T(\omega) \) is the largest nonnegative constant such that

\[
\forall (y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega) \quad C_T(\omega) \| (y^0, y^1) \|^2_{L^2 \times H^{-1}} \leq \int_0^T \int_\omega |y(t, x)|^2 \, dx \, dt
\]

The system is said observable on \([0, T]\) if \( C_T(\omega) > 0 \) (otherwise, \( C_T(\omega) = 0 \)).
Waves propagating in a cavity:

\[ \partial_{tt} y - \Delta y = 0 \]
\[ y(t, \cdot) |_{\partial \Omega} = 0 \]

Observable
\[ y(t, \cdot) |_{\omega} \]

Observability inequality

The observability constant \( C_T(\omega) \) is the largest nonnegative constant such that

\[ \forall (y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega) \quad C_T(\omega) \| (y^0, y^1) \|^2_{L^2 \times H^{-1}} \leq \int_0^T \int_{\omega} |y(t, x)|^2 \, dx \, dt \]

Bardos-Lebeau-Rauch (1992): Observability holds if the pair \((\omega, T)\) satisfies the Geometric Control Condition (GCC) in \( \Omega \):

Every ray of geometrical optics that propagates in \( \Omega \) and is reflected on its boundary \( \partial \Omega \) intersects \( \omega \) in time less than \( T \).
Waves propagating in a cavity:

\[ \partial_{tt} y - \Delta y = 0 \]
\[ y(t, \cdot)|_{\partial\Omega} = 0 \]

Observable
\[ y(t, \cdot)|_{\omega} \]

**Observability inequality**

The observability constant \( C_T(\omega) \) is the largest nonnegative constant such that

\[
\forall (y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega) \quad C_T(\omega)\|(y^0, y^1)\|_{L^2 \times H^{-1}}^2 \leq \int_0^T \int_{\omega} |y(t, x)|^2 \, dx \, dt
\]

**Q:** What is the "best possible" subdomain \( \omega \) of fixed given measure? (say, \( |\omega| = L|\Omega| \) with \( 0 < L < 1 \))

**N.B.:** we want to optimize not only the placement but also the shape of \( \omega \), over all possible measurable subsets. (they do not have a prescribed shape, they are not necessarily BV, etc)
Related problems

1) What is the "best domain" for achieving HUM optimal control?

\[ y_{tt} - \Delta y = \chi_\omega u \]

2) What is the "best domain" domain for stabilization (with localized damping)?

\[ y_{tt} - \Delta y = -k\chi_\omega y_t \]

Existing works by

- P. Hébrard, A. Henrot: theoretical and numerical results in 1D for optimal stabilization.
- A. Münch, P. Pedregal, F. Periago: numerical investigations (fixed initial data).
- S. Cox, P. Freitas, F. Fahroo, K. Ito, ...: variational formulations and numerics.
- M.I. Frecker, C.S. Kubrusly, H. Malebranche, S. Kumar, J.H. Seinfeld, ...: numerical investigations over a finite number of possible initial data.
- ...
The model

Observability inequality

\[ \forall y \text{ solution} \quad C_T(\omega) \|(y(0, \cdot), \partial_t y(0, \cdot))\|_{L^2 \times H^{-1}}^2 \leq \int_0^T \int_\omega |y(t, x)|^2 \, dx \, dt \]

Let \( L \in (0, 1) \) and \( T > 0 \) fixed.

It is a priori natural to model the problem as:

\[
\sup_{\omega \subset \Omega \quad |\omega| = L |\Omega|} C_T(\omega)
\]

with

\[
C_T(\omega) = \inf \left\{ \frac{\int_0^T \int_\omega |y(t, x)|^2 \, dx \, dt}{\|(y(0, \cdot), \partial_t y(0, \cdot))\|_{L^2 \times H^{-1}}^2} \mid (y(0, \cdot), \partial_t y(0, \cdot)) \in L^2(\Omega) \times H^{-1}(\Omega) \setminus \{(0, 0)\} \right\}
\]

BUT...
The model

Observability inequality

\[ \forall y \text{ solution} \quad C_T(\omega) \| (y(0, \cdot), \partial_t y(0, \cdot)) \|_{L^2 \times H^{-1}}^2 \leq \int_0^T \int_\omega |y(t, x)|^2 \, dx \, dt \]

Let \( L \in (0, 1) \) and \( T > 0 \) fixed.

It is \textit{a priori} natural to model the problem as:

\[ \sup_{\omega \subset \Omega} \frac{C_T(\omega)}{|\omega| = L|\Omega|} \]

BUT:

1. Theoretical difficulty due to crossed terms in the spectral expansion (cf Ingham inequalities).

2. In practice: many experiments, many measures. This deterministic constant is \textit{pessimistic}: it gives an account for the \textit{worst case}.

\[ \rightarrow \text{ optimize shape and location of sensors in average, over a large number of measurements} \]

\[ \rightarrow \text{ define an \textit{averaged} observability inequality} \]
Randomized observability constant

Averaging over random initial data:

Randomized observability inequality (wave equation)

\[
C_{T,\text{rand}}(\omega) \|(y(0, \cdot), y_t(0, \cdot))\|_{L^2 \times H^{-1}}^2 \leq \mathbb{E}\left( \int_0^T \int_\omega |y_\nu(t, x)|^2 \, dx \, dt \right)
\]

where

\[
y_\nu(t, x) = \sum_{j=1}^{+\infty} \left( \beta_{1,j}^\nu a_j e^{i\lambda_j t} + \beta_{2,j}^\nu b_j e^{-i\lambda_j t} \right) \phi_j(x)
\]

with \(\beta_{1,j}^\nu, \beta_{2,j}^\nu\) i.i.d. random variables (e.g., Bernoulli, Gaussian) of mean 0


with \((\phi_j)_{j \in \mathbb{N}^*}\) Hilbert basis of eigenfunctions

Randomization
- generates a full measure set of initial data
- does not regularize

Optimal shape and location of sensors
Randomized observability constant

Theorem

\[ C_{T,\text{rand}}(\chi_\omega) = T \inf_{j \in \mathbb{N}^*} \gamma_j \int_\omega \phi_j(x)^2 \, dx \]

with

\[ \gamma_j = \begin{cases} 
1/2 & \text{for the wave equation} \\
1 & \text{for the Schrödinger equation} \\
\frac{e^{2\lambda_j^2 T} - 1}{2\lambda_j^2} & \text{for the heat equation}
\end{cases} \]

with \((\phi_j)_{j \in \mathbb{N}^*}\) a fixed Hilbert basis of eigenfunctions of \(\triangle\)

Remark

There holds \( C_{T,\text{rand}}(\chi_\omega) \geq C_T(\chi_\omega) \).

For the wave equation, the randomized observability constant is a spectral quantity ignoring the rays’ contribution.

\(\mapsto\) spectral criterion = half of the truth!

There are examples where the inequality is strict:

- in 1D: \(\Omega = (0, \pi), \ T \neq k\pi\).
- in multi-D: \(\Omega\) stadium-shaped, \(\omega\) containing the wings.
Randomized observability constant

**Theorem**

\[ \forall \omega \text{ measurable} \]

\[ C_{T, \text{rand}}(\chi_\omega) = T \inf_{j \in \mathbb{N}^*} \gamma_j \int_\omega \phi_j(x)^2 \, dx \]

with

\[ \gamma_j = \begin{cases} 
1/2 & \text{for the wave equation} \\
1 & \text{for the Schrödinger equation} \\
\frac{e^{2\lambda_j^2 T} - 1}{2\lambda_j^2} & \text{for the heat equation}
\end{cases} \]

with \( (\phi_j)_{j \in \mathbb{N}^*} \) a fixed Hilbert basis of eigenfunctions of \( \triangle \)

**Conclusion:** we model the problem as

\[ \sup_{\omega \subseteq \Omega} \inf_{j \in \mathbb{N}^*} \gamma_j \int_{|\omega|=L|\Omega|} \phi_j(x)^2 \, dx \]
To solve the problem, we distinguish between:

- parabolic equations (e.g., heat, Stokes)
- wave or Schrödinger equations

Remarks

- requires some knowledge on the asymptotic behavior of $\phi_j^2$
- $\mu_j = \phi_j^2 \, dx$ is a probability measure
  - $\Rightarrow$ strong difference between $\gamma_j \sim e^{\lambda_j T}$ (parabolic) and $\gamma_j = 1$ (hyperbolic)
Parabolic equations
(e.g.: heat, Stokes, anomalous diffusions)

We assume that $\Omega$ is piecewise $C^1$

**Theorem**
There exists a unique optimal domain $\omega^*$

- Quite difficult proof, requiring in particular: Hartung minimax theorem; fine lower estimates of $\phi_j^2$ by J. Apraiz, L. Escauriaza, G. Wang, C. Zhang (JEMS 2014)
- Algorithmic construction of the best observation set $\omega^*$: to be followed (further)
Wave and Schrödinger equations

Optimal value

Under appropriate spectral assumptions:

\[
\sup_{\omega \subset \Omega} \left\{ \frac{\inf_{j \in \mathbb{N}^*} \int_{\omega} \phi_j(x)^2 \, dx}{|\omega|} \right\} = L
\]

Proof: 1) convexification (relaxation), 2) no-gap (not obvious because not lsc).

Main spectral assumption:

QUE (Quantum Unique Ergodicity): the whole sequence \( \phi_j^2 \, dx \rightharpoonup \frac{dx}{|\Omega|} \) vaguely.

true in 1D, but in multi-D?
Wave and Schrödinger equations

**Optimal value**

Under appropriate spectral assumptions:

\[
\sup_{\omega \subset \Omega} \inf_{j \in \mathbb{N}^*} \int_{\omega} \phi_j(x)^2 \, dx = L
\]

Relationship to quantum chaos theory:

what are the possible (weak) limits of the probability measures \( \mu_j = \phi_j^2 \, dx \)? (quantum limits, or semi-classical measures)

- See also Shnirelman theorem: ergodicity implies Quantum Ergodicity (QE; but possible gap to QUE!)
- If QUE fails, we may have scars
- QUE conjecture (negative curvature)
Wave and Schrödinger equations

Optimal value

Under appropriate spectral assumptions:

$$\sup_{\omega \subset \Omega} \inf_{j \in \mathbb{N}^*} \int_\omega \phi_j(x)^2 \, dx = L$$

Remark: The above result holds true as well in the disk. Hence the spectral assumptions are not sharp.

(proof: requires the knowledge of all quantum limits in the disk, Privat Hillairet Trélat)

$$\mu_{jk} \rightharpoonup \delta_{r=1}$$

(this is one QL: whispering galleries)
Wave and Schrödinger equations

Optimal value

Under appropriate \textit{spectral assumptions}:

\[
\sup_{\omega \subset \Omega} \inf_{j \in \mathbb{N}^*} \int_{\omega} \phi_j(x)^2 \, dx = L
\]

\begin{itemize}
  \item **Supremum reached?** Open problem in general.
    \begin{itemize}
      \item in 1D: reached \iff $L = 1/2$ (infinite number of optimal sets)
      \item in 2D square: reached over Cartesian products \iff $L \in \{1/4, \, 1/2, \, 3/4\}$
    \end{itemize}
  \end{itemize}

Conjecture: Not reached for generic domains $\Omega$ and generic values of $L$. 

Construction of a \textit{maximizing sequence} (by a kind of homogenization)
Following Hébrard-Henrot (SICON 2005), we consider the finite-dimensional spectral approximation:

\[
\sup_{\omega \subset \Omega} \frac{\min_{1 \leq j \leq N} \gamma_j \int_\omega \phi_j^2(x) \, dx}{|\omega| = L|\Omega|}
\]

**Theorem**

The problem has a unique solution \( \omega^N \).

Moreover, \( \omega^N \) is semi-analytic and thus has a finite number of connected components.
Wave and Schrödinger equations

The complexity of $\omega^N$ is increasing with $N$.

Spillover phenomenon: the best domain $\omega^N$ for the $N$ first modes is the worst possible for the $N+1$ first modes.

Parabolic equations

(e.g., heat, Stokes, anomalous diffusions)

Under a slight spectral assumption:
(satisfied, e.g., by $(-\Delta)^\alpha$ with $\alpha > 1/2$)

The sequence of optimal sets $\omega^N$ is stationary:

$$\exists N_0 \mid \forall N \geq N_0 \quad \omega^N = \omega^{N_0} = \omega^*$$

with $\omega^*$ the optimal set for all modes.

In particular, $\omega^*$ is semi-analytic and thus has a finite number of connected components.

$\Omega = (0, \pi)^2$  \hspace{1cm}  $L = 0.2$

4, 25, 100, 500 eigenmodes
Wave and Schrödinger equations

The complexity of $\omega^N$ is increasing with $N$.

Spillover phenomenon: the best domain $\omega^N$ for the $N$ first modes is the worst possible for the $N+1$ first modes.

Parabolic equations

(e.g., heat, Stokes, anomalous diffusions)

Under a slight spectral assumption:

(satisfied, e.g., by $(-\Delta)^\alpha$ with $\alpha > 1/2$)

The sequence of optimal sets $\omega^N$ is stationary:

$$\exists N_0 \mid \forall N \geq N_0 \quad \omega^N = \omega^N_0 = \omega^*$$

with $\omega^*$ the optimal set for all modes.

In particular, $\omega^*$ is semi-analytic and thus has a finite number of connected components.

$\Omega =$ unit disk $\quad L = 0.2$

1, 25, 100, 400 eigenmodes
**Wave and Schrödinger equations**

The complexity of $\omega^N$ is increasing with $N$.

**Spillover phenomenon**: the best domain $\omega^N$ for the $N$ first modes is the worst possible for the $N+1$ first modes.

**Parabolic equations**

(e.g., heat, Stokes, anomalous diffusions)

Under a slight spectral assumption:

(satisfied, e.g., by $(-\Delta)^\alpha$ with $\alpha > 1/2$)

The sequence of optimal sets $\omega^N$ is stationary:

$$\exists N_0 \mid \forall N \geq N_0 \quad \omega^N = \omega^{N_0} = \omega^*$$

with $\omega^*$ the optimal set for all modes.

In particular, $\omega^*$ is semi-analytic and thus has a finite number of connected components.

$\Rightarrow$ no fractal set!

$$\Omega = (0, \pi)^2$$

1, 4, 9, 16, 25, 36 eigenmodes

$L = 0.2$, $T = 0.05$

$\Rightarrow$ optimal thermometer in a square

E. Trélat

Optimal shape and location of sensors
Conclusion and perspectives

- Same kind of analysis for the **optimal design of the control domain**.
- Intimate relations between domain optimization and quantum chaos (**quantum ergodicity properties**).
- Optimal design for **boundary observability** (P. Jounieaux’ PhD):

\[
\sup_{|\omega|=L|\partial \Omega|} \inf_{j \in \mathbb{N}^*} \gamma_j \int_{\Omega} \frac{1}{\lambda_j} \left( \frac{\partial \phi_j}{\partial \nu} \right)^2 \, d\mathcal{H}^{n-1}
\]

- Strategies to **avoid spillover**?
- **Discretization issues**: do the numerical optimal designs converge to the continuous optimal design as the mesh size tends to 0?

---

Y. Privat, E. Trélat, E. Zuazua,

What can be said for the classical (deterministic) observability constant?

A result for the wave observability constant:
(Humbert Privat Trélat, ongoing)

$$\lim_{T \to +\infty} \frac{C_T(\omega)}{T} = \frac{1}{2} \min \left( \inf_{j \in \mathbb{N}^*} \int_\omega \phi_j^2 \, dx, \lim_{T \to +\infty} \inf_{\gamma \text{ ray}} \frac{1}{T} \int_0^T \chi_\omega(\gamma(t)) \, dt \right)$$

Two quantities: 
- spectral
- geometric (rays)

↓
randomized obs. constant
Modeling Solving

E. Trélat

Optimal shape and location of sensors
Remark: another way of arriving at the criterion (wave equation)

Averaging in time:
Time asymptotic observability inequality:

\[ C_\infty(\chi_\omega) \| (y(0, \cdot), y_t(0, \cdot)) \|_{L^2 \times H^{-1}}^2 \leq \lim_{T \to +\infty} \frac{1}{T} \int_0^T \int_\omega |y(t, x)|^2 \, dx \, dt, \]

with

\[ C_\infty(\chi_\omega) = \inf \left\{ \lim_{T \to +\infty} \frac{1}{T} \frac{\int_0^T \int_\omega |y(t, x)|^2 \, dx \, dt}{\| (y(0, \cdot), y_t(0, \cdot)) \|_{L^2 \times H^{-1}}^2} \mid (y(0, \cdot), y_t(0, \cdot)) \in L^2 \times H^{-1} \setminus \{(0, 0)\} \right\}. \]

Theorem

If the eigenvalues of \( \triangle g \) are simple then

\[ C_\infty(\chi_\omega) = \frac{1}{2} \inf_{j \in \mathbb{N}^*} \int_\omega \phi_j(x)^2 \, dx = \frac{1}{2} J(\chi_\omega). \]

Remarks

- \[ C_\infty(\chi_\omega) \leq \frac{1}{2} \inf_{j \in \mathbb{N}^*} \int_\omega \phi_j(x)^2 \, dx. \]
- \[ \limsup_{T \to +\infty} \frac{C_T(\chi_\omega)}{T} \leq C_\infty(\chi_\omega). \] There are examples where the inequality is strict.
A remark for fixed initial data

If we maximize $\omega \mapsto \int_0^T \int_\omega |y(t, x)|^2 \, dx \, dt$ with **fixed initial data**, then, using a decreasing rearrangement argument:

*There always exists (at least) one optimal set $\omega$. The regularity of $\omega$ depends on the initial data: it may be a Cantor set of positive measure, even for $C^\infty$ data.*

→ In our model, we consider an infimum over all initial data.
A remark on the class of subdomains

Let $A > 0$ fixed. If we restrict the search to

$$\{ \omega \subset \Omega \mid |\omega| = L|\Omega| \text{ and } P_{\Omega}(\omega) \leq A \}$$

(perimeter)

or

$$\{ \omega \subset \Omega \mid |\omega| = L|\Omega| \text{ and } \| \chi_{\omega} \|_{BV(\Omega)} \leq A \}$$

(total variation)

or

$$\{ \omega \subset \Omega \mid |\omega| = L|\Omega| \text{ and } \omega \text{ satisfies the } 1/A\text{-cone property} \}$$

or

$\omega$ ranges over some finite-dimensional (or ”compact”) prescribed set...

then there always exists (at least) one optimal set $\omega$.

→ but then...
- the complexity of $\omega$ may increase with $A$
- we want to know if there is a ”very best” set (over all possible measurable)
1. Existence of a maximizer

Ensured if $\mathcal{U}_L$ is replaced with any of the following choices:

- $\mathcal{V}_L = \{\chi_\omega \in \mathcal{U}_L | P_\Omega(\omega) \leq A\}$ (perimeter)
- $\mathcal{V}_L = \{\chi_\omega \in \mathcal{U}_L | \|\chi_\omega\|_{BV(\Omega)} \leq A\}$ (total variation)
- $\mathcal{V}_L = \{\chi_\omega \in \mathcal{U}_L | \omega \text{ satisfies the } 1/A\text{-cone property}\}$

where $A > 0$ is fixed.
2. Weighted observability inequality

\[
C_{T, \sigma}(\chi_\omega) \left( \| (y^0, y^1) \|_{L^2 \times H^{-1}}^2 + \sigma \| y^0 \|_{H^{-1}}^2 \right) \leq \int_0^T \int_\omega |y(t, x)|^2 \, dx \, dt,
\]

where \( \sigma \geq 0 \): weight.

Note that \( C_{T, \sigma}(\chi_\omega) \leq C_T(\chi_\omega) \).

Randomization \( \Rightarrow \) \( 2 \, C_{T, \sigma, \text{rand}}(\chi_\omega) = TJ_\sigma(\chi_\omega) \), where

\[
J_\sigma(\chi_\omega) = \inf_{j \in \mathbb{N}^*} \sigma_j \int_\omega \phi_j(x)^2 \, dx,
\]

with \( \sigma_j = \frac{\lambda_j^2}{\sigma + \lambda_j^2} \).
Remedies (wave and Schrödinger equations)

Theorem

Assume that $L^\infty$-QUE holds. If $\sigma_1 < L < 1$ then there exists $N \in \mathbb{N}^*$ such that

$$\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \sigma_j \int_\omega \phi_j^2 = \max_{\chi_\omega \in \mathcal{U}_L} \inf_{1 \leq j \leq n} \sigma_j \int_\omega \phi_j^2 \leq \sigma_1 < L,$$

for every $n \geq N$. In particular there is a unique solution $\chi_{\omega^N}$. Moreover if $M$ is analytic then $\omega^N$ is semi-analytic and has a finite number of connected components.

- The condition $\sigma_1 < L < 1$ seems optimal (see numerical simulations).
- This result holds as well in any torus, or in the Euclidean $n$-dimensional square for Dirichlet or mixed Dirichlet-Neumann conditions.
Modeling Solving

\[ L = 0.2 \]

\[ L = 0.4 \]

\[ L = 0.6 \]

\[ L = 0.9 \]
An additional remark

Anomalous diffusion equations, Dirichlet: \( \partial_t y + (-\Delta)^\alpha y = 0 \) \((\alpha > 0 \text{ arbitrary})\)
with a surprising result:

In the square \( \Omega = (0, \pi)^2 \), with the usual basis (products of sine): the optimal domain \( \omega^* \) has a finite number of connected components, \( \forall \alpha > 0 \).

In the disk \( \Omega = \{ x \in \mathbb{R}^2 \mid \| x \| < 1 \} \), with the usual basis (Bessel functions), the optimal domain \( \omega^* \) is radial, and
- \( \alpha > 1/2 \implies \omega^* = \text{finite number of concentric rings (and } d(\omega, \partial \Omega) > 0) \)
- \( \alpha < 1/2 \implies \omega^* = \text{infinite number of concentric rings accumulating at } \partial \Omega! \)
  (or \( \alpha = 1/2 \) and \( T \) small enough)

The proof is long and very technical. It uses in particular the knowledge of quantum limits in the disk.

(L. Hillairet, Y. Privat, E.Trélat)
$\Omega = \text{unit disk}$  
1, 4, 9, 16, 25, 36 eigenmodes

$L = 0.2, \ T = 0.05, \ \alpha = 1$
\[ \Omega = \text{unit disk} \quad \text{1, 4, 25, 100, 144, 225 eigenmodes} \]

\[ L = 0.2, \ T = 0.05, \ \alpha = 0.15 \]
### Comparison

<table>
<thead>
<tr>
<th></th>
<th>square</th>
<th>disk</th>
</tr>
</thead>
<tbody>
<tr>
<td>wave or Schrödinger</td>
<td>relaxed solution $a = L$</td>
<td>relaxed solution $a = L$</td>
</tr>
<tr>
<td></td>
<td>$\exists \omega$ for $L \in {\frac{1}{4}, \frac{1}{2}, \frac{3}{4}}$</td>
<td>$\exists \omega$ for $L \in {\frac{1}{4}, \frac{1}{2}, \frac{3}{4}}$</td>
</tr>
<tr>
<td></td>
<td>$\not\exists$ otherwise (conjecture)</td>
<td>$\not\exists$ otherwise (conjecture)</td>
</tr>
<tr>
<td>diffusion $(-\Delta)^\alpha$</td>
<td>$\exists! \omega \ \forall L \ \forall \alpha &gt; 0$</td>
<td>$\exists! \omega$ (radial) $\forall L \ \forall \alpha &gt; 0$</td>
</tr>
<tr>
<td></td>
<td>$#c.c.(\omega) &lt; +\infty$ if $\alpha &gt; 1/2$ then $#c.c.(\omega) &lt; +\infty$</td>
<td>if $\alpha &lt; 1/2$ then $#c.c.(\omega) = +\infty$</td>
</tr>
</tbody>
</table>