Stochastic Closure Schemes for bi-stable Energy Harvesters Excited by Colored Noise

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Bi-stable energy harvester subjected to random excitations

- Robustness under different loads
- Broadband operation
- Suitable for very weak loads
- Straightforward to implement

Walking vibrations

Ocean waves

Challenge: Most sources of energy are neither broadband nor monochromatic

Goal: Model the stochastic dynamics of a strongly nonlinear system involving multiple time scales and correlated excitations
Methods to analyze systems subjected to correlated noise

**Fokker Planck Equation + Filters**
- Model the excitation as filtered white-noise
- Solve the coupled system+filter FP equation
- Very expensive and often unrealistic

**Polynomial Chaos (Wiener, …)**
- Expand excitation and solution in a PC series
- Slow convergence for non-Gaussian responses…
- Little information about non-Gaussian statistics

**Statistical (non) linearization (Booton, Caughey, …)**
- Approximate dynamics by the closest linear system
- Very powerful method for vibrational systems
- Fails for bi-stable (bi-modal) systems
Plan of the presentation – overview of the method

- Overview of statistical linearization methods and their limitations

- The moment-equation-closure minimization method
  - Moment equations expressing two-times statistics
  - Two-times pdf representations and induced closure schemes
  - Simultaneous error minimization for both the moments and the closure

- Representation of the full probability density function

- Application to bistable systems
  - Application to Duffing oscillator excited by correlated noise
  - Application to a bistable electromechanical energy harvester
  - Comparison with Gaussian closure methods
Overview of statistical linearization methods

Consider a nonlinear (SDOF) oscillator of the form:

\[\ddot{x} + \lambda \dot{x} + g(x) = \ddot{y}\]

\[g(x) = k_1 x + k_3 x^3\]

\[y \rightarrow \text{Correlated random excitation}\]

Statistical linearization: substitute the non-linear system by the “closest” linear

\[\ddot{x} + \lambda \dot{x} + k_0 x = \ddot{y}\]

\[E[\Delta^2] = E[(k_0 x - g(x))^2] = \min\]

\[\Rightarrow k_0 = \frac{E[xg(x)]}{E[x^2]}\]
Overview of statistical linearization methods

Step 1: Adoption of a pdf representation (single-time statistics)

\[ k_0 = \frac{E[xg(x)]}{E[x^2]} = \frac{E[k_1x^2 + k_3x^4]}{E[x^2]} = k_1 + 3k_3\sigma_x^2 \]

(Islerlis' Theorem)

Non-Gaussian pdf may also be utilized (statistical non-linearization)

Step 2: Two-times moment equation for the linear system

\[
\ddot{x} + \lambda \dot{x} + k_0 x = \ddot{y}
\]

(WK Theorem)

\[ S_{XX}(\omega) = \frac{-\omega^2}{\left[-\omega^2 + \lambda \omega + k_0\right]^2} S_{YY}(\omega) \]

(Two-time statistics)

\[ \sigma_x^2 = \int_{0}^{\infty} \frac{-\omega^2}{\left[-\omega^2 + \lambda i \omega + k_1 + 3k_3\sigma_x^2\right]^2} S_{YY}(\omega) \]

(Algebraic equation for \( \sigma_x^2 \))
Limitations of statistical linearization methods

1. Moment equations express two-times statistics but adopted pdf representation is for a single-time statistics.

\[ k_0 = \frac{E[xg(x)]}{E[x^2]} \]

*Closure relies on single-time statistics…*

*Important for bi-stable system where we have rich correlation structure*

2. Closure has to be exactly satisfied and all the mismatch is handled by the equation.

\[ \sigma_x^2 = \int_0^\infty \frac{-\omega^2}{-\omega^2 + \lambda i \omega + k_1 + 3k_3 \sigma_x^2} S_{yy}(\omega) \]

*Information obtained by the equation under the condition*

\[ k_0 = \frac{E[xg(x)]}{E[x^2]} \]

*What if the closure condition is not exactly satisfied?*

*bi-stable systems have non-trivial pdf structure*
The moment-equation-closure minimization method

**Step 1:** Develop a pdf representation for two-times statistics

We want this representation to:

i. incorporate specific properties or information about the response pdf (single time statistics) in the statistical steady state

ii. incorporate a given correlation structure between the statistics of the response and the excitation, e.g. Gaussian

iii. have a consistent marginal with the excitation pdf (for the case of the joint response-excitation pdf),

iv. induce a non-Gaussian closure scheme that will be consistent with all the above properties.
The moment-equation-closure minimization method

**Step 1:** Develop a pdf representation for two-times statistics

**Single-time statistics:**

\[ f(x; \gamma) = \frac{1}{\mathcal{F}} \exp \left\{ -\frac{1}{\gamma} \left( \frac{1}{2} k_1 x^2 + \frac{1}{4} k_3 x^4 \right) \right\} \]

*Shape that is consistent with the exact solution of the FP equation but with a free parameter*

**Two-times statistics:**

\begin{align*}
\text{response-excitation pdf} \quad q(x, y) &= \frac{1}{\mathcal{M}} f(x) g(y) e^{c_{xy}} \\
\text{response-response pdf} \quad p(x, z) &= \frac{1}{\mathcal{N}} f(x) f(z) e^{c_{xz}}
\end{align*}

\begin{align*}
&f(x) : \text{marginal for } x(t) \text{ or } x(s) \\
g(y) : \text{marginal for } x(t) \\
&c \quad : \text{depends on } t-s \text{ and expresses degree of correlation}
\end{align*}

Generic non-Gaussian marginals - Gaussian correlation structure
The moment-equation-closure minimization method

**Two-times statistics:**

- **response-excitation pdf** $x(t)y(s)$
  
  $$q(x, y) = \frac{1}{\mathcal{M}} f(x)g(y)e^{cxy}$$

- **response-response pdf** $x(t)x(s)$
  
  $$p(x, z) = \frac{1}{\mathcal{N}} f(x)f(z)e^{czx}$$

Where:

- $f(x)$: marginal for $x(t)$ or $x(s)$
- $g(y)$: marginal for $x(t)$
- $c$: depends on $t$-$s$ & expresses degree of correlation
The moment-equation-closure minimization method

Step 2: Formulation of moment equations for the original system

\[
\ddot{x}(t)y(s) + \lambda \dot{x}(t)y(s) + k_1 x(t)y(s) + k_3 x(t)^3y(s) = \ddot{y}(t)y(s)
\]

\[
\ddot{x}(t)x(s) + \lambda \dot{x}(t)x(s) + k_1 x(t)x(s) + k_3 x(t)^3x(s) = \ddot{y}(t)x(s)
\]

Assuming statistical stationarity: \( \tau = t - s \)

\[
\frac{\partial^2}{\partial \tau^2} C_{xy}(\tau) + \lambda \frac{\partial}{\partial \tau} C_{xy}(\tau) + k_1 C_{xy}(\tau) + k_3 x(t)^3y(s) = \frac{\partial^2}{\partial \tau^2} C_{yy}(\tau),
\]

\[
\frac{\partial^2}{\partial \tau^2} C_{xx}(\tau) + \lambda \frac{\partial}{\partial \tau} C_{xx}(\tau) + k_1 C_{xx}(\tau) + k_3 x(t)^3x(s) = \frac{\partial^2}{\partial \tau^2} C_{xy}(-\tau)
\]

Different equations for \( C_{xx}(\tau) \) and \( C_{xy}(\tau) \)

In statistical (non-) linearization closure is applied directly to the governing eq.

Here we apply closure to the exact two-times moment equations instead…
The moment-equation-closure minimization method

Step 3: *Induced two-times closures for the terms* \( \overline{x(t)^3y(s)} \) *and* \( \overline{x(t)^3x(s)} \)

With some explicit computations using the two-times pdf representations we obtain

**Closure constraint**

\[
\overline{x(t)^3x(s)} = \rho_{x,x} \overline{x(t)x(s)} \quad \rho_{x,x} = \frac{x^4}{x^2} \quad \text{function of } \gamma
\]

Using similar arguments we obtain a closure for \( \overline{x(t)^3y(s)} \)

\[
\rho_{x,y} = \frac{\overline{x^3y}}{\overline{xy}} = \frac{x^4y^2c + \frac{1}{6} (x^6y^4 - 3x^4x^2(y^2)^2) c^3}{x^2y^2c + \frac{1}{6} (x^4y^4 - 3(x^2)^2(y^2)^2) c^3}
\]
The moment-equation-closure minimization method

Substitute the induced two-times closures to the moment equations

\[
\frac{\partial^2}{\partial \tau^2} C_{xy}(\tau) + \lambda \frac{\partial}{\partial \tau} C_{xy}(\tau) + (k_1 + \rho_{x,y} k_3) C_{xy}(\tau) = \frac{\partial^2}{\partial \tau^2} C_{yy}(\tau),
\]

\[
\frac{\partial^2}{\partial \tau^2} C_{xx}(\tau) + \lambda \frac{\partial}{\partial \tau} C_{xx}(\tau) + (k_1 + \rho_{x,x} k_3) C_{xx}(\tau) = \frac{\partial^2}{\partial \tau^2} C_{xy}(-\tau).
\]

We transform the two time equations to spectrum equations

\[
S_{xx}(\omega) = \left| \frac{\omega^4}{\{k_1 + \rho_{x,y} k_3 - \omega^2 + j(\lambda \omega)\}\{k_1 + \rho_{x,x} k_3 - \omega^2 - j(\lambda \omega)\}} \right| S_{yy}(\omega)
\]

From which we obtain the following constraint:

Dynamic constraint

\[
\overline{x^2} = \int_0^\infty \left| \frac{\omega^4}{\{k_1 + \rho_{x,y} k_3 - \omega^2 + j(\lambda \omega)\}\{k_1 + \rho_{x,x} k_3 - \omega^2 - j(\lambda \omega)\}} \right| S_{yy}(\omega) d\omega
\]
The moment-equation-closure minimization method

Step 4: Simultaneous minimization of the two constraints

\[ J(\gamma, \rho_{x,x}) = \left( \overline{x^2} - \int_0^\infty \frac{\omega^4 S_{yy}(\omega)}{\{k_1 + \rho_{x,y}k_3 - \omega^2 + j(\lambda\omega)\}\{k_1 + \rho_{x,x}k_3 - \omega^2 - j(\lambda\omega)\}} \, d\omega \right)^2 + \left( \frac{\overline{x^4}}{\overline{x^2}} - \overline{x^4} \right)^2 \]

Dynamics constraint

Closure constraint

Notes:

- For the case where the closure constraint is exactly satisfied we recover the statistical (non-) linearization method.

- After we obtain the two unknowns we can go back and recover the correlation functions \( C_{xx}(\tau) \) and \( C_{xy}(\tau) \)
The moment-equation-closure minimization method

- Using the values of the correlation functions we can find the constant $c$ to obtain the full joint (two-times) pdf:

**Full pdf representation**

$$f_{x(t)x(t+\tau)y(t+\tau)}(x, z, y) = \frac{1}{\mathcal{R}} f(x; \gamma) f(z; \gamma) g(y) \exp(c_1 x z + c_2 x y + c_3 y z)$$

**Correlation functions from the pdf...**

$$C_{xx}(\tau) = \iiint x z f_{x(t)x(t+\tau)y(t+\tau)}(x, z, y) \, dx \, dy \, dz = c_1 (\bar{x}^2)^2 + \mathcal{O}(c_1^2)$$

$$C_{xy}(\tau) = \iiint x y f_{x(t)x(t+\tau)y(t+\tau)}(x, z, y) \, dx \, dy \, dz = c_2 \bar{x}^2 \bar{y}^2 + \mathcal{O}(c_2^2)$$

$$C_{xy}(0) = \iiint y z f_{x(t)x(t+\tau)y(t+\tau)}(x, z, y) \, dx \, dy \, dz = c_3 \bar{x}^2 \bar{y}^2 + \mathcal{O}(c_3^2)$$
Application to the Duffing equation under correlated excitation

\[ \ddot{x} + \lambda \dot{x} + k_1 x + k_3 x^3 = \dot{y} \]

excitation is a Gaussian with spectrum

\[ S(\omega) = q \frac{1}{\omega^5} \exp\left(-\frac{1}{\omega^4}\right) \]

\[ q = 2 \]
Application to the Duffing equation under correlated excitation

\[ \ddot{x} + \lambda \dot{x} + k_1 x + k_3 x^3 = \ddot{y} \]

excitation is a Gaussian with spectrum

\[ S(\omega) = q \frac{1}{\omega^5} \exp\left(-\frac{1}{\omega^4}\right) \]

\[ \int x(t) x(t+\tau) y(t+\tau) (x, z, y) \]

Monte Carlo Simulation

(a) \( T = 3 \)

(b) \( T = 10 \)

MECM Method
Bistable oscillator coupled to an electromechanical harvester

\[ \ddot{x} + \lambda \dot{x} + k_1 x + k_3 x^3 + \alpha \dot{v} = \ddot{y}, \quad \dot{v} + \beta v = \delta \dot{x} \]

excitation is a Gaussian with spectrum

\[ S(\omega) = q \frac{1}{\omega^5} \exp\left(-\frac{1}{\omega^4}\right) \]
Bistable oscillator coupled to an electromechanical harvester

\[
\ddot{x} + \lambda \dot{x} + k_1 x + k_3 x^3 + \alpha u = \ddot{y}, \quad \text{excitation is a Gaussian with spectrum}
\]

\[
S(\omega) = q \frac{1}{\omega^5} \exp\left(-\frac{1}{\omega^4}\right)
\]

\[C_{xx}\]

\[C_{vv}\]

\[q = 2\]
Bistable oscillator coupled to an electromechanical harvester

\[ \ddot{x} + \lambda \dot{x} + k_1 x + k_3 x^3 + \alpha v = \ddot{y}, \]
\[ \dot{v} + \beta v = \delta \dot{x} \]

Excitation is a Gaussian with spectrum

\[ S(\omega) = q \frac{1}{\omega^5} \exp\left(-\frac{1}{\omega^4}\right) \]

\[ \int x(t)x(t+\tau)y(t+\tau)(x, z, y) \]

Monte Carlo Simulation

MECM Method

(a) \( T = 3 \)

(b) \( T = 10 \)